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Strong Uniqueness, Lipschitz Continuity, and Continuous Selections for Metric Projections in L_1

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The relations between the lower semicontinuity of the metric projection P_G onto a finite-dimensional subspace G of L_1, the Lipschitz continuity of P_G, the existence of continuous selections for P_G, and uniform strong uniqueness of P_G are studied. In particular, the lower semicontinuity of P_G, the Lipschitz continuity of P_G, and the uniform strong uniqueness of P_G are all equivalent. If P_G is lower semicontinuous, then P_G has a Lipschitz continuous selection. Moreover, if G is one-dimensional, P_G has a continuous selection if and only if it has a Lipschitz continuous selection. © 1991 Academic Press, Inc.

1. INTRODUCTION

We will study strong uniqueness, Lipschitz continuous and continuous selections for metric projections in $L_1(T, \mu)$, and some relationships which hold between these properties. Our study reveals that the theory of metric projections in $L_1(T, \mu)$ contrasts dramatically from the theory in $C_0(T)$. Our approach is to study the uniform Hausdorff strong uniqueness of metric projections, since the uniform Hausdorff strong uniqueness implies the Lipschitz continuity of metric projections [25, 26]. This is not surprising, since it is a common practice to prove the pointwise Lipschitz continuity of P_G in $C_0(T)$ by first showing that P_G is strongly unique. There is now a large body of literature that has evolved from the study of strong uniqueness and Lipschitz continuity of metric projections in $C_0(T)$ (see, e.g., [3, 6, 10, 22–24] and references therein).

A particular feature of our approach is that vector measure theory plays an essential role in the proofs of the key results. More specifically, the Liapunov convexity theorem [18, 20] and the Landers connectivity theorem [14] will be used to handle nonatomic measurable sets. Recall that the Liapunov convexity theorem is quite useful for proving results about best L_1-approximations. Two well-known results in L_1 approxima-

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tion theory were given elegant proofs via the Liapunov convexity theorem; one is about the nonexistence of finite-dimensional Chebyshev subspaces on a nonatomic measure space [27, 28], and another is about the equivalence of A-subspaces and finite-dimensional Chebyshev subspaces with respect to varying weights [30].

Before stating the main results in this paper, we define the notation and terminology which will be used. $L_1(T, \mu)$ will denote the Banach space of all integrable functions on the measure space (T, μ) with the norm defined by

$$\|f\| = \int_T |f| \, d\mu \qquad \text{for} \quad f \in L_1(T, \mu).$$

G will always denote a finite-dimensional subspace of $L_1(T, \mu)$. The *metric projection* P_G from $L_1(T, \mu)$ onto G is the set-valued mapping from $L_1(T, \mu)$ onto G defined by

$$P_G(f) = \{g \in G : \|f - g\| = d(f, G)\} \qquad \text{for} \quad f \in L_1(T, \mu),$$

where

$$d(F_1, F_2) := \sup_{g_1 \in F_2} \inf_{g_2 \in F_2} \|g_1 - g_2\| \qquad \text{for} \quad F_1, F_2 \subset L_1(T, \mu).$$

G is called a *Chebyshev subspace* if $P_G(f)$ is a singleton for every f. The Hausdorff metric $H(\cdot, \cdot)$ is defined on the collection of all nonempty closed and bounded subsets $\mathscr{C}(L_1(T, \mu))$ of $L_1(T, \mu)$ by

$$H(F_1, F_2) := \max\{d(F_1, F_2), d(F_2, F_1)\} \qquad \text{for} \quad F_1, F_2 \in \mathscr{C}(L_1(T, \mu)).$$

For $f \in L_1(T, \mu)$, let $Z(f) := \{t \in T : f(t) = 0\}$, $\text{supp}(f) := T \backslash Z(f)$, and $\text{supp}(G) := \bigcup_{g \in G} \text{supp}(g)$. As usual, all subsets of T are only defined up to a set of measure zero. A measurable subset A of T is called an atom if $\mu(A) > 0$ and for any measurable subset B of A, either $\mu(B) = \mu(A)$ or $\mu(B) = 0$. Following [8], we will call a set *unifat* if it is the union of finitely many atoms. A nonzero function $g \in L_1(T, \mu)$ is said to satisfy the *Lazar condition* if whenever $B \subset \text{supp}(g)$ with $\int_B |g| \, d\mu = \|g\|/2$, then either B or $\text{supp}(g) \backslash B$ is a unifat.

Since dim G is finite, recall that P_G is *lower semicontinuous* if and only if

$$\lim_{h \to f} H(P_G(f), P_G(h)) = 0$$

for each $f \in L_1(T, \mu)$. P_G is said to be *Lipschitz continuous* if there is a constant $\lambda > 0$ such that

$$H(P_G(f), P_G(h)) \leqslant \lambda \cdot \|f - h\|$$

for all $f, h \in L_1(T, \mu)$. P_G is said to be *uniformly Hausdorff strongly unique* if there is a constant $\gamma > 0$ such that

$$\|f - g\| \geqslant d(f, G) + \gamma \cdot d(g, P_G(f)),$$

for all $f \in L_1(T, \mu)$, $g \in G$. This is the set-valued generalization of the usual strong uniqueness for Chebyshev sets. A mapping $S\colon L_1(T, \mu) \to G$ is called a *continuous* (*or Lipschitz continuous*) *selection* for P_G if S is continuous (or Lipschitz continuous) and $S(f) \in P_G(f)$ for each $f \in L_1(T, \mu)$.

We can now summarize the main results of this paper. In Section 2 we include some basic facts about $L_1(T, \mu)$-approximation. One interesting case is when $L_1(T, \mu) = l_1(n)$, the n-dimensional Euclidean space with the l_1-norm. There we see that P_G is Lipschitz continuous and has a Lipschitz continuous selection for any subspace G of $l_1(n)$. Moreover, if G is a Chebyshev subspace of $l_1(n)$, then P_G is uniformly strongly unique (cf. Corollary 2.1). In Section 3 Lipschitz continuous metric selections are studied. It turns out that, for one-dimensional subspaces $G = \text{span}\{g\}$, P_G has a Lipschitz continuous selection if g satisfies the Lazar condition (Theorem 3.2). Hence, we deduce from [8] that if $G = \text{span}\{g\}$, then P_G has a Lipschitz continuous selection if and only if P_G has a continuous selection. In Section 4 we show that the elements of $P_G(f)$ are completely determined by their behavior on the atomic part of supp(G), provided that G is a finite-dimensional subspace of $L_1(T, \mu)$ and P_G is lower semicontinuous (Theorem 4.2). Moreover, there is a union A of finitely many atoms in T such that g is a best L_1-approximation to f from G in $L_1(T, \mu)$ if and only if $g|_A$ is a best L_1-approximation to $f|_A$ from $G|_A$ in $L_1(A, \mu)$ for any $f \in L_1(T, \mu)$ (cf. the remark after Lemma 5.1). In the final Section 5 we see that the lower semicontinuity of P_G, the Lipschitz continuity of P_G, and the uniformly Hausdorff strong uniqueness of P_G are all equivalent (Theorem 5.2). In particular, if G is a finite-dimensional Chebyshev subspace of $L_1(T, \mu)$, then P_G is uniformly strongly unique and Lipschitz continuous. Further, if P_G is lower semicontinuous, then P_G has a *Lipschitz* continuous selection (Corollary 5.2). This is a substantially stronger result than can be deduced solely from the Michael selection theorem [22].

We conclude the introduction by mentioning some results in the space $C_0(T)$ which provide a striking contrast to the analogous ones in $L_1(T, \mu)$. (Here T is a locally compact Hausdorff space and $C_0(T)$ is the Banach space of all real continuous functions f on T such that $\{t \in T : |f(t)| \geqslant \varepsilon\}$ is compact for each $\varepsilon > 0$, and $\|f\| = \max_{t \in T} |f(t)|$.) If T is compact and infinite and G is a finite-dimensional Chebyshev subspace of $C_0(T)$, then P_G is Lipschitz continuous if and only if $\dim G = 1$ [2, 4, 5, 16]. If G is a finite-dimensional subspace of $C_0(T)$, then P_G has a Lipschitz continuous selection if and only if P_G is Lipschitz continuous [16]. (In $L_1(T, \mu)$, there exists a finite-dimensional subspace G such that P_G is not lower semi-continuous, but P_G has Lipschitz continuous—even linear—selections (cf. [7, 19])). In $C_0(T)$, there exists a one-dimensional Chebyshev subspace G whose metric projection P_G is not Lipschitz continuous (cf. [16]). (This should be contrasted with Theorem 3.3.)

2. Some Basic Facts about L_1-Approximation

In this section we present some basic facts about L_1-approximation which will be used later in this chapter. Lemmas 2.1–2.5 are known results. Lemmas 2.6 and 2.7 are elementary lemmas about the w^*-topology on $L_\infty(A, \mu)$ for a purely atomic set A. Theorem 2.1 is of interest in its own right. It shows that if M is a subspace of a polyhedral space, then P_M is uniformly Hausdorff strongly unique and Lipschitz continuous. As a consequence, we obtain that if $\operatorname{supp}(G)$ is a unifat, then P_G is uniformly Hausdorff strongly unique and Lipschitz continuous.

First let us recall some known facts about best L_1-approximations.

Lemma 2.1 (Kripke and Rivlin [13]). *Let $f \in L_1(T, \mu)$ and $g \in G$. Then $g \in P_G(f)$ if and only if*

$$\int_{Z(f-g)} |p| \, d\mu \geqslant \int_T p \cdot \operatorname{sign}(f-g) \, d\mu \qquad \text{for all} \quad p \in G.$$

Lemma 2.2 (Phelps [28]). *If $B \cap \operatorname{supp}(G)$ is non-atomic, then there is a mapping $\varphi\colon B \to \{-1, 1\}$ such that $\int_B g \cdot \varphi \, d\mu = 0$ for all $g \in G$.*

Lemma 2.3 (Deutsch, Indumathi, and Schnatz [8]). *Let $g \in L_1(T, \mu) \setminus \{0\}$ and $G = \operatorname{span}\{g\}$. Then P_G has a continuous selection if and only if g satisfies the Lazar condition.*

The special case when $L_1(T, \mu) = l_1$ was proved in a different way by Lazar [15].

Lemma 2.4 (Li [17]). *P_G is lower semicontinuous if and only if $\operatorname{supp}(g_1 - g_2)$ is a unifat for any $f \in L_1(T, \mu)$ and distinct $g_1, g_2 \in P_G(f)$.*

Lemma 2.5. *Let $f \in L_1(T, \mu)$, $g^* \in P_G(f)$, and $g \in G$. Then $g \in P_G(f)$ if and only if g satisfies*

(1) $[f(t) - g(t)][f(t) - g^*(t)] \geqslant 0$ *for $t \in T$, and*

(2) $\int_{Z(f-g^*)} |g - g^*| \, d\mu = \int_T (g - g^*) \cdot \operatorname{sign}(f - g^*) \, d\mu.$

Remark. See Strauss [31], Pinkus [29], and Li [17].

LEMMA 2.6. *Let A be a purely atomic subset of T and let $\{\varphi_j\}$ be a bounded sequence in $L_\infty(A, \mu)$. Then $w^*\text{-}\lim_{j\to\infty} \varphi_j = \varphi$ if and only if $\lim_{j\to\infty} \varphi_j(e) = \varphi(e)$ for all $e \in A$.*

Proof. Suppose $w^*\text{-}\lim_{j\to\infty} \varphi_j = \varphi$. Let $A = \{e_k : k \in I\}$. Then

$$\lim_{j\to\infty} \int_{e_k} \varphi_j \, d\mu = \int_{e_k} \varphi \, d\mu \qquad \text{for} \quad k \in I;$$

i.e.,

$$\lim_{j\to\infty} \varphi_j(e_k) \cdot \mu(e_k) = \varphi(e_k) \cdot \mu(e_k) \qquad \text{for} \quad k \in I, \tag{2.1}$$

which is equivalent to the atomwise convergence of $\{\varphi_j\}$. On the other hand, if (2.1) holds, then

$$\lim_{j\to\infty} \int_A \varphi_j \cdot f \, d\mu = \int_A \varphi \cdot f \, d\mu$$

for $f \in L_1(A, \mu)$ with $\mathrm{supp}(f)$ a unifat. However, the set of all $f \in L_1(A, \mu)$ with $\mathrm{supp}(f)$ a unifat is dense in $L_1(A, \mu)$. Thus, (2.1) implies that $\{\varphi_j\}$ is w^*-convergent to φ. This proves Lemma 2.6. ∎

Next we show that the set of measurable signatures on a purely atomic set A is w^*-compact in $L_\infty(A, \mu)$. Define

$$\Phi := \{\varphi \in L_\infty(A, \mu) : \varphi(e) \in \{-1, 0, 1\} \text{ for } e \in A\}, \tag{2.2}$$

$$\mathscr{X} := \{\varphi \in L_\infty(A, \mu) : \varphi(e) \in \{0, 1\} \text{ for } e \in A\} = \{\chi_B : B \subset A\}. \tag{2.3}$$

Since Φ and $\mathscr{X}$ are closed under the atomwise convergence, by Lemma 2.6, Φ and $\mathscr{X}$ are w^*-closed in $L_\infty(A, \mu)$. By the Alaoglu–Bourbaki theorem [12], Φ and $\mathscr{X}$ are w^*-compact. This proves the following lemma.

LEMMA 2.7. *Φ and $\mathscr{X}$ are w^*-compact for any purely atomic set A.*

Recall that a finite-dimensional normed linear space X is called a polyhedral space, if the unit ball $B(X)$ of X is the convex hull of a finite set [11, 21]. Maserick [21] showed that X is a polyhedral space if and only if its dual X^* is a polyhedral space. Now we want to show that if M is a subspace of a polyhedral space X, then P_M is uniformly Hausdorff strongly unique and Lipschitz continuous. As a consequence, we obtain that P_G is uniformly Hausdorff strongly unique and Lipschitz continuous for any subspace G of $L_1(T, \mu)$, provided $\mathrm{supp}(G)$ is a unifat.

THEOREM 2.1. *Suppose that* X *is a polyhedral space. Then for any subspace* M *of* X, P_M *is uniformly Hausdorff strongly unique; i.e., there exists* $\lambda > 0$ *such that*

$$\|x-g\| \geqslant d(x, M) + \lambda \cdot d(g, P_M(x)) \qquad \text{for} \quad g \in M,\ x \in X. \tag{2.4}$$

Furthermore, P_M *is Lipschitz continuous; i.e., there exists* $c > 0$ *such that*

$$H(P_M(x), P_M(y)) \leqslant c \cdot \|x-y\| \qquad \text{for} \quad x, y \in X, \tag{2.5}$$

where $H(\cdot, \cdot)$ *is the Hausdorff metric.*

Proof. By [21, Theorem 2.7], the dual X^* of X is a polyhedral space. Then the unit ball $B(X^*)$ of X^* is a convex hull of a finite set $\{x_j^*\}_1^r$. Thus, we have

$$\|x\| = \sup\{|x_j^*(x)| : 1 \leqslant j \leqslant r\} \qquad \text{for} \quad x \in X. \tag{2.6}$$

Let $T = \{x_j^*\}_1^r$. Define $\varphi: X \to C(T)$ as follows:

$$\varphi(x)(x_j^*) := x_j^*(x) \qquad \text{for} \quad 1 \leqslant j \leqslant r. \tag{2.7}$$

Let $M_\varphi := \varphi(M)$. Then M_φ is a subspace of $C(T)$. Since for any $g \in M_\varphi$, $T \backslash Z(g)$ is a compact set, a result of Li [16] implies that P_{M_φ} is uniformly Hausdorff strongly unique and Lipschitz continuous. Thus, there exist $\lambda > 0$, $c > 0$ such that

$$\|f-g\| \geqslant d(f, M_\varphi) + \lambda \cdot d(g, P_{M_\varphi}(f)) \qquad \text{for} \quad g \in M_\varphi,\ f \in C(T), \tag{2.8}$$

$$H(P_{M_\varphi}(f), P_{M_\varphi}(h)) \leqslant c \cdot \|f-h\| \qquad \text{for} \quad f, h \in C(T). \tag{2.9}$$

Now, by (2.6) and (2.7), it is easy to verify that $\|x\| = \|\varphi(x)\|$, $d(x, M) = d(\varphi(x), M_\varphi)$, $d(g, P_M(x)) = d(\varphi(g), P_{M_\varphi}(\varphi(x)))$, and $H(P_M(x), P_M(y)) = H(P_{M_\varphi}(\varphi(x)), P_{M_\varphi}(\varphi(y)))$. Hence, (2.4) and (2.5) follow from (2.8) and (2.9). ∎

COROLLARY 2.1. *Suppose that* supp(G) *is a unifat. Then* P_G *is uniformly Hausdorff strongly unique and* P_G *is Lipschitz continuous.*

Proof. Let $A = \text{supp}(G)$ and $X = \{f \in L_1(T, \mu) : T \backslash A \subset Z(f)\}$. Then G is a subspace of the polyhedral space X. By Theorem 2.1, $P_{G \mid X}$ is uniformly Hausdorff strongly unique and Lipschitz continuous; i.e., there exist $\lambda > 0$ and $c > 0$ such that

$$\|f-g\| \geqslant d(f, G) + \lambda \cdot d(g, P_G(f)) \qquad \text{for} \quad g \in G,\ f \in X, \tag{2.10}$$

$$H(P_G(f), P_G(h)) \leqslant c \cdot \|f-h\| \qquad \text{for} \quad f, h \in X. \tag{2.11}$$

Given any $f, h \in L_1(T, \mu)$, let $f_0, h_0 \in X$ be such that $f_0 = f$ on A and $h_0 = h$ on A. Then $P_G(f) = P_G(f_0)$ and $P_G(h) = P_G(h_0)$. Thus, by (2.10), we get

$$\|f - g\| = \int_{T \setminus A} |f| \, d\mu + \|f_0 - g\|$$
$$\geqslant \int_{T \setminus A} |f| \, d\mu + d(f_0, G) + \lambda \cdot d(g, P_G(f_0))$$
$$= d(f, G) + \lambda \cdot d(g, P_G(f)).$$

By (2.11), we deduce that, for $f, h \in L_1(T, \mu)$,

$$H(P_G(f), P_G(h)) = H(P_G(f_0), P_G(h_0)) \leqslant c \cdot \|f_0 - h_0\| \leqslant c \cdot \|f - h\|.$$

Thus, P_G is uniformly Hausdorff strongly unique and Lipschitz continuous. ∎

Remark. A consequence of Corollary 2.1 is a result of Angelos and Schmidt [1], which states that if (T, μ) is a unifat, then P_G is strongly unique at f whenever $P_G(f)$ is a singleton.

3. Lipschitz Continuous Metric Selections and Lazar's Condition

Our main goal in this section is to show that P_G has a continuous selection if and only if P_G has a Lipschitz continuous selection, provided G is a one-dimensional subspace of $L_1(T, \mu)$. Our method is to reduce the problem to the case that $\text{supp}(G)$ is a unifat. More specifically, we will show that if P_G has a continuous selection, then some elements in $P_G(f)$ can be determined by their behavior on a unifat. To do so, we need a formally stronger, but equivalent, version of the Lazar condition which is the key to the reduction procedure mentioned above.

To get the stronger version of the Lazar condition, we need the following corollary of the Liapunov convexity theorem [18, 20, 27].

Lemma 3.1. *If B is a non-atomic set and $\int_B |g| \, d\mu > c \geqslant 0$, then there exist $E \subset B$ such that $\int_E |g| \, d\mu = c$.*

Now we can show that the nonatomic part of $\text{supp}(g)$ is not essential in the Lazar condition. In the sequel, we will denote the atomic part of $\text{supp}(g)$ by $\text{at}(g)$.

Lemma 3.2. *Suppose that $g \in L_1(T, \mu)$ satisfies the Lazar condition. If $B \subset \text{supp}(g)$ and $\int_B |g| \, d\mu > \|g\|/2$, then $\int_{\text{at}(g) \cap B} |g| \, d\mu \geqslant \|g\|/2$.*

Proof. If $\int_{\mathrm{at}(g)\cap B} |g|\,d\mu < \|g\|/2$, then $B\backslash(B\cap \mathrm{at}(g))$ is nonatomic and

$$\int_{B\backslash(B\cap \mathrm{at}(g))} |g|\,d\mu = \int_B |g|\,d\mu - \int_{B\cap \mathrm{at}(g)} |g|\,d\mu$$

$$> \frac{\|g\|}{2} - \int_{B\cap \mathrm{at}(g)} |g|\,d\mu.$$

By Lemma 3.1, there is $E \subset B\backslash \mathrm{at}(g)$ such that

$$0 < \mu(E) < \mu(B\backslash \mathrm{at}(g)), \qquad \text{and} \qquad \int_{E\cup(\mathrm{at}(g)\cap B)} |g|\,d\mu = \frac{\|g\|}{2}.$$

Since $\mu(E) > 0$, $\mu((B\backslash \mathrm{at}(g))\backslash E) > 0$, and E and $(B\backslash \mathrm{at}(g))\backslash E$ are purely nonatomic, we know that $E\cup(\mathrm{at}(g)\cap B)$ and $\mathrm{supp}(g)\backslash\{E\cup(\mathrm{at}(g)\cap B)\}$ both are not unifat, which contradicts the fact that g satisfies the Lazar condition. ▮

Next we show that we can replace the $\mathrm{at}(g)$ in Lemma 3.2 by a unifat.

LEMMA 3.3. *Suppose that g satisfies the Lazar condition. Then there is a unifat set $A \subset \mathrm{supp}(g)$ such that for any $B \subset \mathrm{supp}(g)$ with $\int_B |g| > \|g\|/2$, we have*

$$\int_{B\cap A} |g|\,d\mu \geqslant \frac{\|g\|}{2}.$$

Proof. For convenience, let us denote

$$\mathscr{X} := \{\chi_B : B \subset \mathrm{at}(g)\}, \tag{3.1}$$

$$S(\varphi) := \int_T \varphi\cdot|g|\,d\mu \qquad \text{for} \quad \varphi\in\mathscr{X}. \tag{3.2}$$

It follows from Lemma 2.7 that $\mathscr{X}$ is w^*-compact. Since $S(\cdot)$ is w^*-continuous on $\mathscr{X}$,

$$\mathscr{N} := \left\{\varphi\in\mathscr{X} : S(\varphi) = \frac{\|g\|}{2}\right\} \tag{3.3}$$

is a w^*-compact subset of $\mathscr{X}$. Let $\chi_B \in \mathscr{N}$. We discuss the following two cases:

(1) B is a unifat:

Then

$$V(\chi_B) := \{\varphi\in\mathscr{X} : \varphi(e) = \chi_B(e) \text{ for } e\in B\}$$

is a (relatively) w^*-open neighborhood of χ_B in $\mathscr{X}$. Obviously,

$$S(\varphi)=\int_T \varphi\cdot|g|\,d\mu>\int_T \chi_B\cdot|g|\,d\mu=\frac{\|g\|}{2} \qquad \text{for} \quad \varphi\in V(\chi_B)\backslash\{\chi_B\}. \tag{3.4}$$

(2) B is not a unifat:

Then, since g satisfies the Lazar condition, $\operatorname{supp}(g)\backslash B$ is a unifat. Thus,

$$V(\chi_B):=\{\varphi\in\mathscr{X}:\varphi(e)=\chi_B(e) \text{ for } e\in\operatorname{supp}(g)\backslash B\}$$

is a (relatively) w^*-open neighborhood of χ_B in $\mathscr{X}$. Obviously,

$$S(\varphi)=\int_T \varphi\cdot|g|\,d\mu<\int_T \chi_B\cdot|g|\,d\mu=\frac{\|g\|}{2} \qquad \text{for} \quad \varphi\in V(\chi_B)\backslash\{\chi_B\}. \tag{3.5}$$

Moreover, $\operatorname{supp}(g)$ is a unifat; i.e., $\operatorname{supp}(g)=\operatorname{at}(g)$.

If $\operatorname{at}(g)$ is a unifat, let $A=\operatorname{at}(g)$. Then Lemma 3.3 follows from Lemma 3.2. Thus we may assume

$$\operatorname{at}(g)=\{e_i\}_1^\infty.$$

If Lemma 3.3 fails to be true, then there are $B_k\subset\operatorname{supp}(g)$ such that for $k\geqslant 1$,

$$S(\chi_{B_k})>\frac{\|g\|}{2}, \qquad \text{and} \tag{3.6}$$

$$S(\chi_{B_k\cap\{e_i\}_1^k})<\frac{\|g\|}{2}. \tag{3.7}$$

Since $\mathscr{X}$ is w^*-compact, we may assume that for some $B\subset\operatorname{at}(g)$,

$$w^*\text{-}\lim_{k\to\infty}\chi_{B_k\cap\{e_i\}_1^k}=\chi_B. \tag{3.8}$$

It follows from Lemma 2.6 that (3.8) implies

$$w^*\text{-}\lim_{k\to\infty}\chi_{B_k\cap\{e_i\}_1^\infty}=\chi_B. \tag{3.9}$$

By (3.6) and Lemma 3.2, we obtain

$$S(\chi_{B_k\cap\{e_i\}_1^\infty})\geqslant\frac{\|g\|}{2}. \tag{3.10}$$

It follows from (3.7)–(3.10) that

$$S(\chi_B)=\frac{\|g\|}{2}; \tag{3.11}$$

i.e., $\chi_B \in \mathcal{N}$. Recalling that $V(\chi_B)$ (defined in cases (1) and (2) above) is a (relatively) w^*-open neighborhood of χ_B, it follows that there exists n such that

$$\chi_{B_n \cap \{e_i\}_1^\infty}, \qquad \chi_{B_n \cap \{e_i\}_1^n} \in V(\chi_B). \tag{3.12}$$

If B is a unifat, then (3.4) and (3.12) contradict (3.7). Otherwise, it follows from (3.5), (3.10), and (3.12) that

$$\chi_B=\chi_{B_n \cap \{e_i\}_1^\infty}. \tag{3.13}$$

Since $\operatorname{supp}(g)=\operatorname{at}(g)=\{e_j\}_1^\infty$ in this case, (3.13) implies $\chi_B=\chi_{B_n}$, which contradicts (3.11) and (3.6). The contradiction proves Lemma 3.3. ∎

Remark. The proof of Lemma 3.3 implies a formally stronger version of the Lazar condition. In fact, (3.4) and (3.5) imply that each φ in $\mathcal{N}$ is an isolated point. Since $\mathcal{N}$ is w^*-compact, $\mathcal{N}$ must be a finite set. Let A be the union of all unifats B whose characteristic function $\chi_B \in \mathcal{N}$. Then A is still a unifat. Let $B \subset \operatorname{supp}(g)$ be such that $\int_B |g|\, d\mu=\|g\|/2$. Then either B or $\operatorname{supp}(g)\backslash B$ is a unifat whose characteristic function is in $\mathcal{N}$. Therefore, either $B\subset A$ or $\operatorname{supp}(g)\backslash B \subset A$. This proves the following corollary.

COROLLARY 3.1. *Let $g \in L_1(T,\mu)\backslash\{0\}$. Then g satisfies the Lazar condition if and only if there is a unifat $A \subset \operatorname{supp}(g)$ such that either $B \subset A$ or $\operatorname{supp}(g)\backslash B \subset A$ whenever B is a subset of $\operatorname{supp}(g)$ with $\int_B |g|\, d\mu=\|g\|/2$.*

By Lemma 3.3, we can show that if g satisfies the Lazar condition and $G=\operatorname{span}\{g\}$, then some elements in $P_G(f)$ are completely determined by their behavior on the unifat A in Lemma 3.3.

THEOREM 3.1. *Suppose that $g \in L_1(T,\mu)\backslash\{0\}$ satisfies the Lazar condition and $G=\operatorname{span}\{g\}$. Then there is a unifat $A \subset \operatorname{supp}(g)$ such that*

$$P_G(f)\supset\{p\in G : p|_B \in P_{G|_B}(f|_B)\} \qquad \textit{for all} \quad f\in L_1(T,\mu),\ B\supset A. \tag{3.14}$$

Proof. Let A be the same unifat as in Lemma 3.3. Then for any $B \supset A$ and any $E \subset \operatorname{supp}(g)$ with $\int_E |g| > \|g\|/2$, we have

$$\int_{E\cap B} |g|\, d\mu \geqslant \int_{E\cap A} |g|\, d\mu \geqslant \frac{\|g\|}{2}. \tag{3.15}$$

Fix $f \in L_1(T, \mu)$ and $B \supset A$. Let $p \in G$ be such that $p|_B \in P_{G|_B}(f|_B)$. By Lemma 2.1,

$$\int_B q \cdot \operatorname{sign}(f-p)\, d\mu \leqslant \int_{Z(f-p) \cap B} |q|\, d\mu \qquad \text{for} \quad q \in G,$$

which is equivalent to

$$\left| \int_B g \cdot \operatorname{sign}(f-p)\, d\mu \right| \leqslant \int_{Z(f-p) \cap B} |g|\, d\mu. \tag{3.16}$$

Let $A_i = \{t \in T : (-1)^i \cdot g(t) \cdot \operatorname{sign}(f(t) - p(t)) > 0\}$, $i = 1, 2$. If $\int_{A_i} |g|\, d\mu > \|g\|/2$ for some i, then, by (3.15), we get

$$\int_{A_i \cap B} |g|\, d\mu \geqslant \frac{\|g\|}{2}. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\frac{\|g\|}{2} \leqslant \int_{A_i \cap B} |g|\, d\mu \leqslant \int_{B \setminus A_i} |g|\, d\mu. \tag{3.18}$$

But then

$$\begin{aligned} \|g\| = \int_T |g|\, d\mu &= \int_{A_i} |g|\, d\mu + \int_{T \setminus A_i} |g|\, d\mu \\ &\geqslant \int_{A_i} |g|\, d\mu + \int_{B \setminus A_i} |g|\, d\mu \\ &> \frac{\|g\|}{2} + \frac{\|g\|}{2} = \|g\|, \end{aligned}$$

which is absurd. Thus we must have

$$\int_{A_i} |g|\, d\mu \leqslant \frac{\|g\|}{2} \qquad \text{for} \quad i = 1, 2.$$

Then

$$\begin{aligned} \int_{A_1} |g|\, d\mu \leqslant \frac{\|g\|}{2} &\leqslant \int_{T \setminus A_1} |g|\, d\mu \\ &= \int_{A_2} |g|\, d\mu + \int_{Z(f-p)} |g|\, d\mu. \end{aligned}$$

Interchanging the roles of A_1 and A_2, we deduce

$$\left|\int_{A_1} |g|\, d\mu - \int_{A_2} |g|\, d\mu\right| \leqslant \int_{Z(f-p)} |g|\, d\mu,$$

which is equivalent to

$$\left|\int_T q \cdot \operatorname{sign}(f-p)\, d\mu\right| \leqslant \int_{Z(f-p)} |q|\, d\mu \qquad \text{for} \quad q \in G.$$

By Lemma 2.1, $p \in P_G(f)$. Hence, (3.14) holds. ∎

Next we give an application of Corollary 2.1, which shows that if we can reduce the L_1-approximation to the L_1-approximation on a unifat of T, then the metric projection has a Lipschitz continuous selection.

LEMMA 3.4. *Suppose that there is a unifat A such that*

$$P_G(f) \supset \{g \in G : g\,|_A \in P_{G\,|_A}(f\,|_A)\} \qquad \textit{for all} \quad f \in L_1(T, \mu).$$

Then P_G has a Lipschitz continuous selection.

Proof. If $g \in G$ and $\int_A |g|\, d\mu = 0$, then $0 \in P_{G\,|_A}(g\,|_A)$; i.e.,

$$P_G(g) = \{g\} \supset \{0\},$$

which implies $g = 0$. Hence, $\int_A |g|\, d\mu$ for $g \in G$ defines a norm on G. Thus, there is a constant $\alpha > 0$ such that

$$\|g\| \leqslant \alpha \int_A |g|\, d\mu \qquad \text{for} \quad g \in G. \tag{3.19}$$

Let

$$P_G(f, A) := \{g \in G : g\,|_A \in P_{G\,|_A}(f\,|_A)\} \qquad \text{for} \quad f \in L_1(T, \mu).$$

Then $P_G(f, A)\,|_A = P_{G\,|_A}(f\,|_A)$ for $f \in L_1(T, \mu)$. By Corollary 2.1, there is $\beta > 0$ such that, for $f, h \in L_1(T, \mu)$,

$$H(P_G(f, A)\,|_A, P_G(h, A)\,|_A) \leqslant \beta \int_A |f-h|\, d\mu \leqslant \beta \cdot \|f-h\|. \tag{3.20}$$

By (3.19), we obtain

$$\begin{aligned} &H(P_G(f, A), P_G(h, A)) \\ &\qquad \leqslant \alpha H(P_G(f, A)\,|_A, P_G(h, A)\,|_A) \qquad \text{for} \quad f, h \in L_1(T, \mu). \end{aligned} \tag{3.21}$$

It follows from (3.20) and (3.21) that $P_G(\cdot, A)$ is Lipschitz continuous. Obviously, the set $P_G(f, A)$ is convex and compact for every $f \in L_1(T, \mu)$. By a result of Deutsch, Li, and Park [9, Proposition 2.3], $P_G(\cdot, A)$ has a Lipschitz continuous selection σ. Since $\sigma(f) \in P_G(f, A) \subset P_G(f)$, σ is also a Lipschitz continuous selection for P_G. ∎

As a consequence of Lemma 3.4 and Theorem 3.1, we have the following theorem.

THEOREM 3.2. *Suppose that* $g \in L_1(T, \mu) \backslash \{0\}$ *satisfies the Lazar condition and* $G = \operatorname{span}\{g\}$. *Then* P_G *has a Lipschitz continuous selection.*

Finally, let us summarize the results proved in this section, together with the known Lemma 2.3.

THEOREM 3.3. *Let* $g \in L_1(T, \mu) \backslash \{0\}$ *and* $G = \operatorname{span}\{g\}$. *Then the following are equivalent*:

(1) P_G *has a continuous selection*;

(2) P_G *has a Lipschitz continuous selection*;

(3) g *satisfies the Lazar condition*;

(4) *There is a unifat* $A \subset \operatorname{supp}(g)$ *such that either* $B \subset A$ *or* $\operatorname{supp}(g) \backslash B \subset A$ *whenever* $B \subset \operatorname{supp}(g)$ *with* $\int_B |g| \, d\mu = \|g\|/2$.

4. BEST APPROXIMATION ON THE ATOMIC PART OF supp(G)

In this section we show that the elements in $P_G(f)$ are completely determined by their behavior on the atomic part at(G) of supp(G) (cf. Theorem 4.2), provided P_G is lower semicontinuous.

For convenience, let us use the following notations:

$$\Phi := \{\varphi \in L_\infty(\operatorname{at}(G), \mu) : \varphi(e) \in \{-1, 0, 1\} \text{ for } e \in \operatorname{at}(G)\}, \tag{4.1}$$

$$Q(\varphi, g, E) := \int_{Z(\varphi)} |g| \, d\mu - \int_{\operatorname{at}(G)} \varphi \cdot g \, d\mu - \int_E |g| \, d\mu, \tag{4.2}$$

$$\mathcal{N}(\varphi, E) := \{g \in G : Q(\varphi, g, E) = 0\}. \tag{4.3}$$

Our intention is to show that $Q(\varphi, g, T \backslash \operatorname{at}(G)) \geqslant 0$ for all $g \in G$, provided $Q(\varphi, g, \varnothing) \geqslant 0$ for all $g \in G$ (Theorem 4.1). This will be used to prove that $g|_{\operatorname{at}(G)}$ is a best L_1-approximation to $f|_{\operatorname{at}(G)}$ from $G|_{\operatorname{at}(G)}$ if and only if $g \in P_G(f)$ (Theorem 4.2). To do so, we need to show that elements in $\mathcal{N}(\varphi, E)$ have unifat supports (Lemma 4.1) and $Q(\varphi, g, E) \leqslant \lambda(\varphi, E) \cdot \int_{T \backslash (\operatorname{at}(G) \cup E)} |g| \, d\mu$ for $g \in G$ (a corollary of Lemma 4.2).

LEMMA 4.1. *Suppose that* P_G *is lower semicontinuous. If* $\varphi \in \Phi$ *and* $E \subset T \backslash \text{at}(G)$ *are such that* $Q(\varphi, p, E) \geqslant 0$ *for all* $p \in G$, *then* $\text{supp}(g)$ *is a unifat for each* $g \in \mathcal{N}(\varphi, E) \backslash \{0\}$.

Proof. By Lemma 2.2, there is a mapping $\varphi_1 : T \backslash (\text{at}(G) \cup E) \to \{-1, 1\}$ such that

$$\int_{T \backslash (\text{at}(G) \cup E)} p \cdot \varphi_1 \, d\mu = 0 \qquad \text{for all} \quad p \in G. \tag{4.4}$$

Suppose $g \in \mathcal{N}(\varphi, E) \backslash \{0\}$. Let $\{g_i\}_1^n$ be a basis of G. Define

$$h = |g| + \sum_{j=1}^{n} |g_j|, \qquad \text{and} \qquad f(t) := \begin{cases} h(t)\, \varphi(t), & t \in \text{at}(G) \\ h(t) \,\text{sign}(g(t)), & t \in E \\ h(t)\, \varphi_1(t), & t \in T \backslash (\text{at}(G) \cup E). \end{cases}$$

From the definition of f, we deduce

$$f(t) \cdot (f(t) - g(t)) \geqslant 0 \qquad \text{for} \quad t \in T. \tag{4.5}$$

From (4.4), the hypothesis that $Q(\varphi, g, E) = 0$, and the definition of f, we deduce that

$$\int_T g \cdot \text{sign}(f) \, d\mu = \int_{Z(f)} |g| \, d\mu. \tag{4.6}$$

Now, by (4.4) and $Q(\varphi, p, E) \geqslant 0$, it is not difficult to verify that

$$\int_{Z(f)} |p| \, d\mu \geqslant \int_T p \cdot \text{sign}(f) \, d\mu \qquad \text{for} \quad p \in G.$$

By Lemma 2.1, $0 \in P_G(f)$. From (4.5), (4.6), and Lemma 2.5 we know that $g \in P_G(f)$. Since P_G is lower semicontinuous, by Lemma 2.4, $\text{supp}(g) = \text{supp}(g - 0)$ is a unifat. ∎

Remark. If $\text{supp}(g)$ is a unifat for some $g \in G \backslash \{0\}$, then

$$\delta(g) := \inf\{|g(e)| \; \mu(e) : e \in \text{supp}(g)\} > 0. \tag{4.7}$$

Therefore, under the assumption of Lemma 4.1,

$$V_{\mathcal{N}(\varphi, E)} := \bigcup_{g \in S^1(G) \cap \mathcal{N}(\varphi, E)} B(g, \delta(g)) \tag{4.8}$$

is a neighborhood of $S^1(G) \cap \mathcal{N}(\varphi, E)$ in G, where $S^1(G)$ denotes the unit sphere of G and $B(g, \varepsilon) := \{p \in G : \|g - p\| < \varepsilon\}$ is the ball in G of radius ε and centered at g.

LEMMA 4.2. *Let* $\varphi \in \Phi$ *and* $E \subset \operatorname{supp}(G)\backslash \operatorname{at}(G)$ *be such that* $Q(\varphi, p, E) \geqslant 0$ *for every* $p \in G$. *If* P_G *is lower semicontinuous, then* $V_{\mathcal{N}(\varphi, E)}$ *is a neighborhood of* $S^1(G) \cap \mathcal{N}(\varphi, E)$. *Moreover,*

$$\lambda(\varphi, E) := \inf_{g \in S^1(G)\backslash \mathcal{N}(\varphi, E)} \frac{Q(\varphi, g, E)}{d(g, \mathcal{N}(\varphi, E))}$$

$$= \min_{g \in S^1(G)\backslash V_{\mathcal{N}(\varphi, E)}} \frac{Q(\varphi, g, E)}{d(g, \mathcal{N}(\varphi, E))} > 0. \tag{4.9}$$

Proof. By Lemma 4.1 and the remark before Lemma 4.2, $V_{\mathcal{N}(\varphi, E)}$ is a neighborhood of $S^1(G) \cap \mathcal{N}(\varphi, E)$. Now we claim that for any $g \in V_{\mathcal{N}(\varphi, E)} \cap S^1(G)\backslash \mathcal{N}(\varphi, E)$, there exists $g^* \in V_{\mathcal{N}(\varphi, E)} \cap S^1(G)$ such that

$$d(g^*, \mathcal{N}(\varphi, E)) \geqslant 2 \cdot d(g, \mathcal{N}(\varphi, E)), \qquad \text{and}$$

$$\frac{Q(\varphi, g^*, E)}{d(g^*, \mathcal{N}(\varphi, E))} \leqslant \frac{Q(\varphi, g, E)}{d(g, \mathcal{N}(\varphi, E))}.$$

In fact, let $g \in V_{\mathcal{N}(\varphi, \mathscr{E})} \cap S^1(G)\backslash \mathcal{N}(\varphi, E)$. Then there exists $p \in S^1(G) \cap \mathcal{N}(\varphi, E)$ such that $\|g-p\| < \frac{1}{2} \cdot \delta(p)$. Thus,

$$|g(e)-p(e)|\, \mu(e) \leqslant \|g-p\| < \tfrac{1}{2}\delta(p) \leqslant \tfrac{1}{2}\, |p(e)|\, \mu(e) \qquad \text{for} \quad e \in \operatorname{supp}(p),$$

which implies

$$|g(t) - \tfrac{1}{2}p(t)| = |g(t)| - \tfrac{1}{2}\, |p(t)| \qquad \text{for} \quad t \in T.$$

Thus,

$$\int_{Z(\varphi)} \left| g - \frac{1}{2} \cdot p \right| d\mu = \int_{Z(\varphi)} |g|\, d\mu - \frac{1}{2} \cdot \int_{Z(\varphi)} |p|\, d\mu, \tag{4.10}$$

$$\int_{E} \left| g - \frac{1}{2} \cdot p \right| d\mu = \int_{E} |g|\, d\mu - \frac{1}{2} \cdot \int_{E} |p|\, d\mu, \qquad \text{and} \tag{4.11}$$

$$\|g - \tfrac{1}{2} \cdot p\| = \|g\| - \tfrac{1}{2} \cdot \|p\| = \tfrac{1}{2}. \tag{4.12}$$

Let $g^* = 2 \cdot g - p$. Then (4.12) implies $g^* \in S^1(G)$. By Lemma 4.1, $\operatorname{supp}(q)$ is a unifat for $q \in \mathcal{N}(\varphi, E)$. Since $Q(\varphi, \cdot, E)$ is positively homogeneous, satisfies the triangle inequality for elements in G with unifat supports, and is nonnegative on G by hypothesis, $\mathcal{N}(\varphi, E)$ is a closed convex cone. Therefore, $\frac{1}{2} \cdot p + \mathcal{N}(\varphi, E) \subset \mathcal{N}(\varphi, E)$ and $\frac{1}{2}\mathcal{N}(\varphi, E) = \mathcal{N}(\varphi, E)$. Hence

$$\begin{aligned} d(g^*, \mathcal{N}(\varphi, E)) &= d(2 \cdot (g - \tfrac{1}{2} \cdot p), \mathcal{N}(\varphi, E)) \\ &= 2 \cdot d(g - \tfrac{1}{2} \cdot p, \tfrac{1}{2} \cdot \mathcal{N}(\varphi, E)) \\ &= 2 \cdot d(g - \tfrac{1}{2} \cdot p, \mathcal{N}(\varphi, E)) \\ &= 2 \cdot d(g, \tfrac{1}{2} \cdot p + \mathcal{N}(\varphi, E)) \\ &\geqslant 2 \cdot d(g, \mathcal{N}(\varphi, E)). \end{aligned} \tag{4.13}$$

Furthermore, (4.10), (4.11), and (4.13) imply

$$\begin{aligned} \frac{Q(\varphi, g^*, E)}{d(g^*, \mathcal{N}(\varphi, E))} &\leqslant \frac{Q(\varphi, g^*, E)}{2 \cdot d(g, \mathcal{N}(\varphi, E))} = \frac{2 \cdot Q(\varphi, g - (1/2) \cdot p, E)}{2 \cdot d(g, \mathcal{N}(\varphi, E))} \\ &= \frac{Q(\varphi, g, E) - (1/2) \cdot Q(\varphi, p, E)}{d(g, \mathcal{N}(\varphi, E))} = \frac{Q(\varphi, g, E)}{d(g, \mathcal{N}(\varphi, E))}. \end{aligned}$$

This proves our claim.

Since $Q(\varphi, \cdot, E)$ and $d(\cdot, \mathcal{N}(\varphi, E))$ are continuous positive functions on the compact set $S^1(G) \backslash V_{\mathcal{N}(\varphi, E)}$,

$$\min_{g \in S^1(G) \backslash V_{\mathcal{N}(\varphi, E)}} \frac{Q(\varphi, g, E)}{d(g, \mathcal{N}(\varphi, E))} > 0.$$

Therefore, it suffices to show that the equality in (4.9) holds. Assume the contrary that the equality in (4.9) does not hold. Then there exists $g_0 \in S^1(G) \backslash \mathcal{N}(\varphi, E)$ such that

$$\frac{Q(\varphi, g_0, E)}{d(g_0, \mathcal{N}(\varphi, E))} < \min_{g \in S^1(G) \backslash V_{\mathcal{N}(\varphi, E)}} \frac{Q(\varphi, g, E)}{d(g, \mathcal{N}(\varphi, E))}. \tag{4.14}$$

Then $g_0 \in V_{\mathcal{N}(\varphi, E)}$. By applying the previous claim inductively, we can get a sequence $\{g_i\}_1^\infty \subset V_{\mathcal{N}(\varphi, E)} \cap S^1(G)$ such that for $i \geqslant 1$,

$$d(g_i, \mathcal{N}(\varphi, E)) \geqslant 2 \cdot d(g_{i-1}, \mathcal{N}(\varphi, E)) \geqslant 2^i \cdot d(g_0, \mathcal{N}(\varphi, E)) \quad \text{and} \tag{4.15}$$

$$\begin{aligned} \frac{Q(\varphi, g_i, E)}{d(g_i, \mathcal{N}(\varphi, E))} &\leqslant \frac{Q(\varphi, g_{i-1}, E)}{d(g_{i-1}, \mathcal{N}(\varphi, E))} \leqslant \frac{Q(\varphi, g_0, E)}{d(g_0, \mathcal{N}(\varphi, E))} \\ &< \min_{g \in S^1(G) \backslash V_{\mathcal{N}(\varphi, E)}} \frac{Q(\varphi, g, E)}{d(g, \mathcal{N}(\varphi, E))}. \end{aligned} \tag{4.16}$$

Since $0 \in \mathcal{N}(\varphi, E)$, $d(g_i, \mathcal{N}(\varphi, E)) \leqslant \|g_i\| = 1$. Thus, (4.15) implies that $2^i \cdot d(g_0, \mathcal{N}(\varphi, E)) \leqslant 1$ for $i \geqslant 1$. Therefore $d(g_0, \mathcal{N}(\varphi, E)) = 0$, which implies $g_0 \in \mathcal{N}(\varphi, E)$. This contradiction completes the proof of Lemma 4.2. ∎

Remark. Since $\mathcal{N}(\varphi, E)$ is a convex cone under the hypothesis of Lemma 4.2, $g \in G \backslash \mathcal{N}(\varphi, E)$ implies $g^* := g/\|g\| \in S^1(G) \backslash \mathcal{N}(\varphi, E)$. Thus, by (4.9) and $\|g\| \cdot Q(\varphi, g^*, E) = Q(\varphi, g, E)$, we get

$$\begin{aligned} Q(\varphi, g, E) &= \|g\| \cdot Q(\varphi, g^*, E) \geqslant \lambda(\varphi, E) \cdot \|g\| \cdot d(g^*, \mathcal{N}(\varphi, E)) \\ &= \lambda(\varphi, E) \cdot d(g, \mathcal{N}(\varphi, E)). \end{aligned}$$

Since $\operatorname{supp}(p) \subset \operatorname{at}(G)$ for $p \in \mathcal{N}(\varphi, E) \backslash \{0\}$,

$$d(g, \mathcal{N}(\varphi, E)) \geqslant \int_{T \backslash (E \cup \operatorname{at}(G))} |g| \, d\mu.$$

This proves the following corollary of Lemma 4.2.

COROLLARY 4.1. *Under the assumption of Lemma* 4.2, *we have* $\lambda(\varphi, E) > 0$ *and*

$$Q(\varphi, g, E) \geqslant \lambda(\varphi, E) \cdot \int_{T \backslash (E \cup \operatorname{at}(G))} |g| \, d\mu \qquad \text{for} \quad g \in G. \tag{4.17}$$

The following result about the range of non-atomic vector measures was proved by Landers [14: Corollary 6]. It will be used in the proof of Theorem 4.1.

LEMMA 4.3. *Let τ be a non-atomic measure defined on a σ-algebra Σ with values in a Banach space X. Then $\tau(\Sigma) \subset X$ is arcwise connected.*

Now we can show that the non-atomic part of $\operatorname{supp}(G)$ is not essential for the best L_1-approximations.

THEOREM 4.1. *Suppose that P_G is lower semicontinuous. If $\varphi \in \Phi$ is such that $Q(\varphi, g, \varnothing) \geqslant 0$ for every $g \in G$, then $Q(\varphi, g, T \backslash \operatorname{at}(G)) \geqslant 0$ for every $g \in G$.*

Proof. Let

$$\mathcal{A} = \{E \subset \operatorname{supp}(G) \backslash \operatorname{at}(G) : Q(\varphi, g, E) \geqslant 0 \text{ for } g \in G\}.$$

Define a partial order on $\mathcal{A}$ by

$$E_1 \leqslant E_2 \qquad \text{if} \qquad E_1 \subset E_2.$$

Let $\mathcal{F}$ be a chain in $\mathcal{A}$. Then $\mathcal{F} = \{E_\alpha : \alpha \in I\}$, where I is a well-ordered set [12]. Define

$$I_k = \{\alpha \in I : \mu_G(E_\alpha \backslash E_\beta) \geqslant 1/k \text{ for every } \beta < \alpha\} \qquad \text{for} \quad k \geqslant 1,$$

where

$$\mu_G(E) := \int_E \sum_{i=1}^{n} |g_i| \, d\mu,$$

and $\{g_i\}_1^n$ is a basis of G.

Since $\mu_G(\text{supp}(G)) < \infty$ and $\mathscr{F}$ is a chain, one can verify that I_k is a finite set. Let $J = \bigcup_{k=1}^{\infty} I_k$. Then J is a countable set. Define

$$E = \bigcup_{\alpha \in J} E_\alpha.$$

Then E is a measurable subset of $\text{supp}(G) \backslash \text{at}(G)$. We claim that

$$\mu(E_\alpha \backslash E) = 0 \qquad \text{for} \quad \alpha \in I.$$

In fact, let

$$I^* = \{\alpha : \mu(E_\alpha \backslash E) > 0\}.$$

If $I^* \neq \varnothing$, since I is well-ordered, there is a minimal index $\alpha^* \in I^*$. If there is a $k \geqslant 1$ such that

$$\mu_G(E_{\alpha^*} \backslash E_\beta) \geqslant 1/k \qquad \text{for} \quad \beta < \alpha^*,$$

then $\alpha^* \in I_k \subset J$, which implies $E_{\alpha^*} \subset E$ and $\mu(E_{\alpha^*} \backslash E) = 0$. This is impossible. Thus, for any $k \geqslant 1$, there exists a $\beta = \beta_k < \alpha^*$ such that

$$\mu_G(E_{\alpha^*} \backslash E_\beta) < 1/k.$$

Since α^* is the minimal index in I^*, $\beta \notin I^*$. Thus, $\mu(E_\beta \backslash E) = 0$, which implies

$$\mu_G(E_\beta \backslash E) = 0.$$

Therefore, we have

$$\mu_G(E_{\alpha^*} \backslash E) \leqslant \mu_G(E_{\alpha^*} \backslash E_\beta) + \mu_G(E_\beta \backslash E) < 1/k.$$

Since $k \geqslant 1$ is arbitrary, we get $\mu_G(E_{\alpha^*} \backslash E) = 0$. Since E_{α^*} is a subset of $\text{supp}(G)$, we have $\mu(E_{\alpha^*} \backslash E) = 0$, which contradicts $\alpha^* \in I^*$. The contradiction proves our claim.

Next we claim that $E \in \mathscr{A}$. In fact, rewrite $E = \bigcup_{j=1}^{\infty} E_j$. Since $\mathscr{F}$ is a chain, $\bigcup_{j=1}^{n} E_j = E_{j_n}$ for some $1 \leqslant j_n \leqslant n$. Therefore,

$$\lim_{n \to \infty} \int_{E_{j_n}} |g| \, d\mu = \int_E |g| \, d\mu \qquad \text{for} \quad g \in G,$$

which implies

$$Q(\varphi, g, E) = \lim_{n \to \infty} Q(\varphi, g, E_{j_n}) \geqslant 0 \qquad \text{for} \quad g \in G.$$

Thus, $E \in \mathscr{A}$. This proves the second claim.

The above two claims imply that every chain in $\mathscr{A}$ has an upper bound. By Zorn's lemma, there is a maximal element $E \in \mathscr{A}$. We will show that $E = \text{supp}(G) \backslash \text{at}(G)$ and hence

$$Q(\varphi, g, T \backslash \text{at}(G)) = Q(\varphi, g, \text{supp}(G) \backslash \text{at}(G)) \geqslant 0 \qquad \text{for every} \quad g \in G,$$

which will complete the proof.

By Corollary 4.1, there is a constant $\alpha > 0$ such that

$$Q(\varphi, g, E) \geqslant \alpha \cdot \int_{T \backslash (\text{at}(G) \cup E)} |g| \, d\mu \qquad \text{for} \quad g \in G. \tag{4.18}$$

We may assume $\alpha < 1$. Let $l^\infty(G)$ be the Banach space of all real bounded functions x on G with the supremum norm $\|x\| = \sup\{|x(g)| : g \in G\}$. Define

$$\tau(B) := \left\{ \frac{\int_B |g| \, d\mu}{Q(\varphi, g, E)} \right\}_{g \in G} \qquad \text{for} \quad B \in \mathscr{B},$$

where $0/0 := 0$ and $\mathscr{B}$ is the σ-algebra generated by the measurable subsets of the set $\text{supp}(G) \backslash (\text{at}(G) \cup E)$. Then $\tau \colon \mathscr{B} \to l^\infty(G)$ is a countably additive nonatomic vector measure. If $E \neq \text{supp}(G) \backslash \text{at}(G)$, then there is $g^* \in G$ such that

$$\int_{\text{supp}(G) \backslash (\text{at}(G) \cup E)} |g^*| \, d\mu > 0.$$

It follows from this and (4.18) that

$$\|\tau(\text{supp}(G) \backslash (\text{at}(G) \cup E))\| > 1.$$

Thus, by Lemma 4.3, there is $E_1 \subset \text{supp}(G) \backslash (\text{at}(G) \cup E)$ such that $\|\tau(E_1)\| = 1$. Thus

$$\frac{\int_{E_1} |g| \, d\mu}{Q(\varphi, g, E)} \leqslant 1$$

for every $g \in G$ implies that

$$Q(\varphi, g, E \cup E_1) = Q(\varphi, g, E) - \int_{E_1} |g| \, d\mu \geqslant 0.$$

That is, $E \cup E_1 \in \mathscr{A}$. Since $\mu(E_1) > 0$, $E \cup E_1 \neq E$, which implies $(E \cup E_1) > E$. This contradicts the fact that E is maximal in $\mathscr{A}$. This contradiction completes the proof of Theorem 4.1. ▮

THEOREM 4.2. *If P_G is lower semicontinuous, then*

$$P_G(f) = \{g \in G : g\,|_{\mathrm{at}(G)} \in P_{G\,|_{\mathrm{at}(G)}}(f\,|_{\mathrm{at}(G)})\} \qquad \textit{for every} \quad f \in L_1(T, \mu).$$

Proof. Denote

$$P_G(f, \mathrm{at}(G)) := \{g \in G : g\,|_{\mathrm{at}(G)} \in P_{G\,|_{\mathrm{at}(G)}}(f\,|_{\mathrm{at}(G)})\}.$$

Let $g \in P_G(f, \mathrm{at}(G))$. Then, by Lemma 2.1, for $\varphi := \mathrm{sign}(f-g)|_{\mathrm{at}(G)} \in \Phi$, $Q(\varphi, p, \varnothing) \geqslant 0$ for $p \in G$. By Theorem 4.1, we obtain that for $p \in G$,

$$\begin{aligned}\int_{Z(f-g)} |p|\, d\mu &\geqslant \int_{Z(f-g) \cap \mathrm{at}(G)} |p|\, d\mu \\ &\geqslant \int_{\mathrm{at}(G)} p \cdot \mathrm{sign}(f-g)\, d\mu + \int_{T \setminus \mathrm{at}(G)} |p|\, d\mu \\ &\geqslant \int_T p \cdot \mathrm{sign}(f-g)\, d\mu.\end{aligned}$$

By Lemma 2.1, $g \in P_G(f)$. Thus,

$$P_G(f, \mathrm{at}(G)) \subset P_G(f) \qquad \text{for} \quad f \in L_1(T, \mu). \tag{4.19}$$

On the other hand, choose $g^* \in P_G(f, \mathrm{at}(G)) \subset P_G(f)$. Fix $g \in P_G(f)$. Since P_G is lower semicontinuous, by Lemma 2.4, $\mathrm{supp}(g - g^*) \subset \mathrm{at}(G)$. Since $g, g^* \in P_G(f)$, by Lemma 2.5, we have

$$(f(t) - g(t)) \cdot (f(t) - g^*(t)) \geqslant 0 \qquad \text{for} \quad t \in T, \qquad \text{and} \tag{4.20}$$

$$\int_{Z(f-g^*)} |g - g^*|\, d\mu = \int_T (g - g^*) \cdot \mathrm{sign}(f - g^*)\, d\mu, \tag{4.21}$$

Since $\mathrm{supp}(g - g^*) \subset \mathrm{at}(G)$, (4.21) implies

$$\int_{Z(f-g^*) \cap \mathrm{at}(G)} |g - g^*|\, d\mu = \int_{\mathrm{at}(G)} (g - g^*) \cdot \mathrm{sign}(f - g^*)\, d\mu. \tag{4.22}$$

Since $g^* \in P_G(f, \mathrm{at}(G))$, by (4.20), (4.22), and Lemma 2.5, we know $g \in P_G(f, \mathrm{at}(G))$. Thus,

$$P_G(f) \subset P_G(f, \mathrm{at}(G)) \qquad \text{for} \quad f \in L_1(T, \mu). \tag{4.23}$$

(4.19) and (4.23) imply $P_G(f) = P_G(f, \mathrm{at}(G))$. This completes the proof of Theorem 4.2. ▮

5. Strong Uniqueness and Lipschitz Continuity

In this section we show that if P_G is lower semicontinuous, then P_G is uniformly Hausdorff strongly unique and Lipschitz continuous. As a consequence we obtain that if G is a finite-dimensional Chebyshev subspace of $L_1(T, \mu)$, then P_G is uniformly strongly unique and Lipschitz continuous.

To do so, we first need to show that the constant $\lambda(\varphi, E)$ in Lemma 4.2 is independent of φ and E.

Lemma 5.1. *Suppose that P_G is lower semicontinuous. Then there is $\lambda > 0$ such that for all $f \in L_1(T, \mu)$ and $g \in P_G(f)$,*

$$\int_{Z(f-g)} |p| \, d\mu - \int_T p \cdot \text{sign}(f-g) \, d\mu \geqslant \lambda \cdot d(p, G_0) \qquad \textit{for} \quad p \in G, \tag{5.1}$$

where $G_0 := \{p \in G : \text{supp}(p) \text{ is a unifat}\}$.

Proof. First recall the notations used in Section 4:

$$\Phi := \{\varphi \in L_\infty(\text{at}(G), \mu) : \varphi(e) \in \{-1, 0, 1\} \text{ for } e \in \text{at}(G)\}, \tag{5.2}$$

$$Q(\varphi, g, E) := \int_{Z(\varphi)} |g| \, d\mu - \int_{\text{at}(G)} \varphi \cdot g \, d\mu - \int_E |g| \, d\mu. \tag{5.3}$$

Moreover, denote

$$\Phi_0 := \{\varphi \in \Phi : Q(\varphi, g, \varnothing) \geqslant 0 \text{ for } g \in G\}. \tag{5.4}$$

By Lemma 2.6, $w^*\text{-lim } \varphi_j = \varphi$ implies $w^*\text{-lim } |\varphi_j| = |\varphi|$. Since

$$\int_{Z(\varphi)} |g| \, d\mu = \int_{\text{at}(G)} (1 - |\varphi|) \cdot |g| \, d\mu,$$

$Q(\varphi, g, \varnothing)$ is w^*-continuous for $\varphi \in \Phi$. Since Φ is w^*-compact (cf. Lemma 2.7), Φ_0 is a w^*-compact subset of Φ. It follows from Theorem 4.1 that

$$Q(\varphi, g, T \backslash \text{at}(G)) \geqslant 0 \qquad \text{for every} \quad g \in G, \ \varphi \in \Phi_0. \tag{5.5}$$

Since $\text{supp}(G_0)$ is a unifat,

$$\begin{aligned} W(\varphi) &:= \{\psi \in \Phi_0 : |\psi(e) - \varphi(e)| < \tfrac{1}{2} \text{ for } e \in \text{supp}(G_0)\} \\ &= \{\psi \in \Phi_0 : \psi(e) = \varphi(e) \text{ for } e \in \text{supp}(G_0)\} \end{aligned} \tag{5.6}$$

is a w^*-neighborhood of $\varphi \in \Phi_0$. Since Φ_0 is w^*-compact, there exist $\{\varphi_j\}_1^n \subset \Phi_0$ such that

$$\Phi_0 = \bigcup_{j=1}^{n} W(\varphi_j). \tag{5.7}$$

Let

$$\mathcal{N}(\varphi) := \mathcal{N}(\varphi, T\backslash \mathrm{at}(G)) = \{g \in G : Q(\varphi, g, T\backslash \mathrm{at}(G)) = 0\} \qquad \text{for} \quad \varphi \in \Phi_0. \tag{5.8}$$

By Lemma 4.1, $\mathcal{N}(\varphi) \subset G_0$ for $\varphi \in \Phi_0$. Thus, for $\varphi \in \Phi_0$,

$$\mathcal{N}(\varphi) = \left\{ g \in G_0 : \int_{\mathrm{supp}(G_0) \cap Z(\varphi)} |g| \, d\mu - \int_{\mathrm{supp}(G_0)} g \cdot \varphi \, d\mu = 0 \right\}. \tag{5.9}$$

Therefore, for $\psi \in W(\varphi)$,

$$\begin{aligned} \mathcal{N}(\psi) &= \left\{ g \in G_0 : \int_{\mathrm{supp}(G_0) \cap Z(\psi)} |g| \, d\mu - \int_{\mathrm{supp}(G_0)} g \cdot \psi \, d\mu = 0 \right\} \\ &= \left\{ g \in G_0 : \int_{\mathrm{supp}(G_0) \cap Z(\varphi)} |g| \, d\mu - \int_{\mathrm{supp}(G_0)} g \cdot \varphi \, d\mu = 0 \right\} = \mathcal{N}(\varphi). \end{aligned} \tag{5.10}$$

Let $V_{\mathcal{N}(\varphi)}$ be the neighborhood of $\mathcal{N}(\varphi) \cap S^1(G)$ in G as defined in (4.8); i.e.,

$$V_{\mathcal{N}(\varphi)} := \bigcup_{g \in S^1(G) \cap \mathcal{N}(\varphi)} B(g, \delta(g)), \tag{5.11}$$

where $S^1(G)$ denotes the unit sphere of G, $B(g, \varepsilon) := \{p \in G : \|g - p\| < \varepsilon\}$ is the ball of radius ε and centered at g in G, and

$$\delta(g) := \inf\{|g(e)| \cdot \mu(e) : e \in \mathrm{supp}(g)\} > 0.$$

Then (5.10) and (5.11) imply

$$V_{\mathcal{N}(\psi)} = V_{\mathcal{N}(\varphi)} \qquad \text{for} \quad \psi \in W(\varphi). \tag{5.12}$$

Hence, by Lemma 4.2, (5.7), (5.12), and $\mathcal{N}(\varphi) \subset G_0$, we obtain

$$
\begin{aligned}
&\inf_{\varphi \in \Phi_0,\, g \in S^1(G)\setminus G_0} \frac{Q(\varphi, g, T\setminus \mathrm{at}(G))}{d(g, G_0)} \\
&= \inf_{\varphi \in \Phi_0} \inf_{g \in S^1(G)\setminus G_0} \frac{Q(\varphi, g, T\setminus \mathrm{at}(G))}{d(g, G_0)} \\
&\geqslant \inf_{\varphi \in \Phi_0} \inf_{g \in S^1(G)\setminus \mathcal{N}(\varphi)} \frac{Q(\varphi, g, T\setminus \mathrm{at}(G))}{d(g, \mathcal{N}(\varphi))} \\
&\geqslant \inf_{1 \leqslant j \leqslant n} \inf_{\varphi \in W(\varphi_j)} \inf_{g \in S^1(G)\setminus V_{\mathcal{N}(\varphi)}} \frac{Q(\varphi, g, T\setminus \mathrm{at}(G))}{d(g, \mathcal{N}(\varphi))} \\
&\geqslant \inf_{1 \leqslant j \leqslant n} \inf_{\varphi \in W(\varphi_j)} \inf_{g \in S^1(G)\setminus V_{\mathcal{N}(\varphi_j)}} \frac{Q(\varphi, g, T\setminus \mathrm{at}(G))}{d(g, \mathcal{N}(\varphi_j))} \qquad (5.13)
\end{aligned}
$$

By (5.6), we know that $W(\varphi_j)$ is also w^*-compact. By (5.10), $Q(\varphi, g, T\setminus \mathrm{at}(G)) = 0$ for $\varphi \in W(\varphi_j)$ implies $g \in \mathcal{N}(\varphi) = \mathcal{N}(\varphi_j)$. Thus, $Q(\varphi, g, T\setminus \mathrm{at}(G))$ and $d(g, \mathcal{N}(\varphi_j))$ are continuous positive functions of (φ, g) on the compact product set $W(\varphi_j) \times (S^1(G)\setminus \mathcal{N}(\varphi_j))$. Therefore,

$$\inf_{\varphi \in W(\varphi_j)} \inf_{g \in S^1(G)\setminus V_{\mathcal{N}(\varphi_j)}} \frac{Q(\varphi, g, T\setminus \mathrm{at}(G))}{d(g, \mathcal{N}(\varphi_j))} > 0 \qquad \text{for} \quad 1 \leqslant j \leqslant n. \qquad (5.14)$$

It follows from (5.13) and (5.14) that

$$\lambda := \inf_{\varphi \in \Phi_0,\, g \in S^1(G)\setminus G_0} \frac{Q(\varphi, g, T\setminus \mathrm{at}(G))}{d(g, G_0)} > 0.$$

Since $Q(\varphi, g, T\setminus \mathrm{at}(G))$ and $d(g, G_0)$ are positive homogeneous with respect to g, it is easy to see that

$$Q(\varphi, g, T\setminus \mathrm{at}(G)) \geqslant \lambda \cdot d(g, G_0) \qquad \text{for} \quad g \in G,\ \varphi \in \Phi_0. \qquad (5.15)$$

Finally, let $f \in L_1(T, \mu)$ and $g \in P_G(f)$. Then, by Theorem 4.2 and Lemma 2.1, $\varphi := \mathrm{sign}(f-g)\,|_{\mathrm{at}(G)} \in \Phi_0$. Thus, by (5.15),

$$
\begin{aligned}
\int_{Z(f-g)} |p|\, d\mu &\geqslant \int_{Z(f-g) \cap \mathrm{at}(G)} |p|\, d\mu = \int_{Z(\varphi)} |p|\, d\mu \\
&\geqslant \int_{\mathrm{at}(G)} p \cdot \varphi\, d\mu + \int_{T\setminus \mathrm{at}(G)} |p|\, d\mu + \lambda \cdot d(p, G_0) \\
&\geqslant \int_T p \cdot \mathrm{sign}(f-g)\, d\mu + \lambda \cdot d(p, G_0)
\end{aligned}
$$

for all $p \in G$. This completes the proof of Lemma 5.1. ∎

Remark. Let $A_0 = \text{supp}(G_0)$. Then $d(g, G_0) \geqslant \int_{T\setminus A_0} |g|\, d\mu$. Therefore, (5.15) implies

$$\int_{Z(\varphi)} |g|\, d\mu - \int_{\text{at}(G)} g \cdot \varphi\, d\mu - \int_{T\setminus \text{at}(G)} |g|\, d\mu \geqslant \lambda \cdot \int_{T\setminus A_0} |g|\, d\mu. \tag{5.16}$$

Suppose $A_0 = \{e_1, ..., e_n\}$ and $\text{at}(G) = \{e_k\}_1^\infty$. Then there exists $m \geqslant n$ such that

$$\int_{\text{at}(G)\setminus \{e_k\}_1^m} |g|\, d\mu \leqslant \frac{\lambda}{4} \cdot \int_{T\setminus A_0} |g|\, d\mu \qquad \text{for} \quad g \in G. \tag{5.17}$$

It follows from (5.16) and (5.17) that

$$\int_{Z(\varphi) \cap \{e_k\}_1^m} |g|\, d\mu - \int_{\{e_k\}_1^m} g \cdot \varphi\, d\mu - \int_{T\setminus \{e_k\}_1^m} |g|\, d\mu \geqslant \frac{\lambda}{4} \cdot \int_{T\setminus \{e_k\}_1^m} |g|\, d\mu. \tag{5.18}$$

Let $A = \{e_k\}_1^m$. Then, by (5.18) and a similar argument to that in Theorem 4.2, we can prove

$$P_G(f) = \{g \in G : g\,|_A \in P_{G\,|_A}(f\,|_A)\} \qquad \text{for} \quad f \in L_1(T, \mu).$$

THEOREM 5.1. *Suppose P_G is lower semicontinuous. Then P_G is uniformly Hausdorff strongly unique; i.e., there is $\beta > 0$ such that*

$$\|f - g\| \geqslant d(f, G) + \beta \cdot d(g, P_G(f)) \qquad \textit{for} \quad f \in L_1(T, \mu),\ g \in G.$$

Proof. Let $G_0 := \{g \in G : \text{supp}(g) \text{ is a unifat}\}$. Since $\text{supp}(G_0)$ is a unifat, by Corollary 2.1, there is $\rho > 0$ such that

$$\|f - g\| \geqslant d(f, G_0) + \rho \cdot d(g, P_{G_0}(f)) \qquad \text{for} \quad f \in L_1(T, \mu),\ g \in G_0. \tag{5.19}$$

For any $f \in L_1(T, \mu)$ and $g \in G$, let $g^* \in P_G(f)$ be such that $\|g - g^*\| = d(g, P_G(f))$. By Lemma 5.1,

$$\begin{aligned}
\|f - g\| &\geqslant \int_{Z(f-g^*)} |f - g|\, d\mu + \int_T (f - g) \cdot \text{sign}(f - g^*)\, d\mu \\
&= \int_{Z(f-g^*)} |g - g^*|\, d\mu - \int_T (g - g^*) \cdot \text{sign}(f - g^*)\, d\mu + \|f - g^*\| \\
&\geqslant \lambda \cdot d(g - g^*, G_0) + \|f - g^*\| \\
&= d(f, G) + \lambda \cdot d(g - g^*, G_0).
\end{aligned} \tag{5.20}$$

Let $p^* \in G_0$ be such that $d(g-g^*, G_0) = \|g-g^*-p^*\|$. Then, by (5.19),

$$\begin{aligned}\|f-g\| &= \|f-g^*-p^*+g^*+p^*-g\| \\ &\geqslant \|f-g^*-p^*\| - d(g-g^*, G_0) \\ &\geqslant d(f-g^*, G_0) + \rho \cdot d(p^*, P_{G_0}(f-g^*)) - d(g-g^*, G_0) \\ &\geqslant d(f, G) + \rho \cdot d(g-g^*, P_{G_0}(f-g^*)) - (1+\rho) \cdot d(g-g^*, G_0).\end{aligned} \tag{5.21}$$

Let

$$\beta := \min\left\{\frac{\rho}{2}, \frac{\rho \cdot \lambda}{2 \cdot (1+\rho)}\right\} > 0.$$

Since $0 \in P_G(f-g^*)$, $d(f-g^*, G_0) = d(f-g^*, G)$. Thus, $P_G(f-g^*) \supset P_{G_0}(f-g^*)$. If

$$\rho \cdot d(g, P_G(f)) > 2 \cdot (1+\rho) \cdot d(g-g^*, G_0).$$

then, by (5.21), we have

$$\begin{aligned}\|f-g\| &\geqslant d(f, G) + \rho \cdot d(g-g^*, P_G(f-g^*)) - (1+\rho) \cdot d(g-g^*, G_0) \\ &= d(f, G) + \rho \cdot d(g, P_G(f)) - (1+\rho) \cdot d(g-g^*, G_0) \\ &> d(f, G) + \frac{\rho}{2} \cdot d(g, P_G(f)) \geqslant d(f. G) + \beta \cdot d(g, P_G(f)).\end{aligned}$$

Otherwise, by (5.20), we get

$$\|f-g\| \geqslant d(f, G) + \frac{\rho \cdot \lambda}{2 \cdot (1+\rho)} \cdot d(g, P_G(f)) \geqslant d(f, G) + \beta \cdot d(g, P_G(f)).$$

Thus, P_G is uniformly Hausdorff strongly unique. ∎

It follows from Park's result [25, 26] that the uniform Hausdorff strong uniqueness of P_G implies the Lipschitz continuity of P_G. Thus, by Theorem 5.1, we have the following characterization of lower semicontinuity of P_G.

THEOREM 5.2. *The following statements are equivalent*:

(1) P_G *is lower semicontinuous*;

(2) P_G *is uniformly Hausdorff strongly unique*;

(3) P_G *is Lipschitz continuous.*

COROLLARY 5.1. *If G is a finite-dimensional Chebyshev subspace of $L_1(T, \mu)$, then P_G is uniformly strongly unique and Lipschitz continuous.*

Remark. This result may seem surprising since for a finite-dimensional Chebyshev subspace M of $C(T)$ with compact infinite T, P_M is Lipschitz continuous if and only if $\dim M = 1$.

We know that if P_G is Lipschitz continuous, then it has a Lipschitz continuous selection [9]. Thus, another easy consequence of Theorem 5.2 is the following stronger version of the Michael selection theorem [22].

COROLLARY 5.2. *If P_G is lower semicontinuous, then P_G has a Lipschitz continuous selection.*

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